5/12/16
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Παράδειγμα: Να λυθεί η εξίσωση:

• $y''' + y' = 2 + \sin x$ $\quad y_0(0) = 0,\ y'(0) = 1,\ y''(0) = -1$

↑ μη ομογενής

$y''' + y' = 0$

$p(\lambda) = \lambda^3 + \lambda = \lambda(\lambda^2 + 1)$ $\quad p(\lambda) = 0 \Rightarrow \lambda = 0,\ \lambda = \pm i$

$B.\Sigma.\Pi = \{e^{0x}, \cos x, \sin x\} = \{1, \cos x, \sin x\}$ (Δεν χρειάζεται να δείξω ότι είναι γρ. ανεξ.)

↱ $y''' + y' = 2$ → βρίσκω τις μερικές λύσεις ξεχωριστά

$\rightsquigarrow y_{p_1}(x) = 2x$

↳ $y''' + y' = \sin x$

$y''' + y' = e^{ix}$

$y = ze^{ix}$

$y_{p_2}(x) = \frac{1}{2} x\sin x$

$\Rightarrow y_p(x) = 2x - \frac{1}{2} x\sin x$

Το σύνολο των λύσεων της εξίσωσης

$y(x) = c_1 + c_2\cos x + c_3 \sin x + 2x - \frac{1}{2}x\sin x$

$y'(x) = \dots \Rightarrow y(0) = 0 \quad c_1 + c_2 = 0$

$y'''(x) = \dots \Rightarrow y'(0) = 1 \quad c_3 + 2 = 1$

$y''(0) = -1 \quad -c_2 - 1 = -1$

$c_1 = c_2 = 0,\ c_3 = -1$

Άσκηση $y'' + y = 3x^2 - 4\sin x$, $\quad y(0) = 0$, $\quad y'(0) = 1$

$y'' + y = 0$

$p(\lambda) = \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$ $\quad$ $B.\Sigma.\Pi = \{\cos x, \sin x\}$

• $y'' + y = 3x^2$

για $y = ax^2 + \beta x + \gamma$, έχουμε $2a + ax^2 + \beta + \gamma = 3x^2$

$\boxed{a = 3}$ $\boxed{\beta = 0}$ $\boxed{2a + \gamma = 0 \Rightarrow \gamma = -6}$

Συνεπώς $y_{p_1}(x) = 3x^2 - 6$

$y'' + y = -4\sin x$

$y'' + y = -4e^{ix}$ $\quad$ Θέτω $y = ze^{ix}$

$y'' = z''e^{ix} + 2iz'e^{ix} - \cancel{ze^{ix}} + \cancel{ze^{ix}} = -4e^{ix}$

$z'' + 2iz' = -4 \quad \rightsquigarrow z' = \frac{-4}{2i} = 2i$

$z(x) = 2ix + \cancel{c}$

$y_{p_2}(x) = \operatorname{Im}[ze^{ix}] = \operatorname{Im}(2ix(\cos x + i\sin x)) = 2x\cos x$

$y_p(x) = y_{p_1}(x) + y_{p_2}(x) = 3x^2 - 6 + 2x\cos x$

Οι λύσεις της εξίσωσης: $y(x) = c_1\cos x + c_2\sin x + 3x^2 - 6 + 2x\cos x$

$y'(x) = -c_1\sin x + c_2\cos x + 6x + 2\cos x - 2x\sin x$

$\left. \begin{array}{l} y(0) = 0 \\ y'(0) = 1 \end{array} \right| \Rightarrow \begin{array}{l} c_1 = 6 \\ c_2 = -2 \end{array}$

B-40 $a, b, c, k > 0$, $bk \neq ak^2 + c$

όλες οι λύσεις της $ay'' + by' + cy = e^{-kx}$, $x \in \mathbb{R}$

τείνουν προς το 0 για $x \to \infty$

$ay'' + by' + cy = 0$

$p(\lambda) = a\lambda^2 + b\lambda + c$ ~~[illegible]~~

$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2c}$

$1^{η}$ περίπτωση $b^2 - 4ac > 0 \rightarrow \lambda_1, \lambda_2 \in \mathbb{R}$

$\lambda_1 \neq \lambda_2$ $\quad \{\lambda_1 < \lambda_2 < 0\}$

? δεν κατάλαβα γιατί

$\lambda_1 + \lambda_2 = -b/a$

$\rho_1\rho_2 = c/a$

$\text{ΘΣΛ} = \{e^{\lambda_1 x}, e^{\lambda_2 x}, x \in \mathbb{R}\}$

$w(x) = \begin{vmatrix} e^{\lambda_1 x} & e^{\lambda_2 x} \\ \lambda_1 e^{\lambda_1 x} & \lambda_2 e^{\lambda_2 x} \end{vmatrix} = e^{\lambda_1 x} \cdot e^{\lambda_2 x} \begin{vmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{vmatrix} = e^{(\lambda_1 + \lambda_2)x}(\lambda_2 - \lambda_1)$

$w_1(x) = \begin{vmatrix} 0 & e^{\lambda_2 x} \\ 1 & \lambda_2 e^{\lambda_2 x} \end{vmatrix} = -e^{\lambda_2 x}$

$w_2(x) = \begin{vmatrix} e^{\lambda_1 x} & 0 \\ \lambda_1 e^{\lambda_1 x} & 1 \end{vmatrix} = e^{\lambda_1 x}$

$$y_p(x) = y_1(x)\int_0^x \frac{w_1(s)}{W(s)}\frac{e^{-ks}}{a}ds + y_2(x)\int_0^x \frac{w_2(s)}{W(s)}\frac{e^{-ks}}{a}ds$$

$$= e^{\lambda_1 x}\int_0^x \frac{e^{-\lambda_2 s}}{(\lambda_2-\lambda_1)e^{(\lambda_1+\lambda_2)s}}\frac{e^{-ks}}{a}ds + e^{\lambda_2 x}\int_0^x \frac{e^{\lambda_1 s}}{(\lambda_2-\lambda_1)e^{(\lambda_1+\lambda_2)s}}\frac{e^{-ks}}{a}ds$$

$$= \frac{1}{(\lambda_2-\lambda_1)a}\left[e^{\lambda_1 x}\int_0^x \frac{e^{-ks}}{e^{\lambda_1 s}} + e^{\lambda_2 x}\int_0^x \frac{e^{-ks}}{e^{\lambda_2 s}}ds\right]$$

$$= \frac{1}{(\lambda_2-\lambda_1)a}\left[-e^{\lambda_1 x}\int_0^x e^{(-k-\lambda_1)s}ds + e^{\lambda_2 x}\int_0^x e^{(-k-\lambda_2)s}ds\right]$$

$$= \frac{1}{(\lambda_2-\lambda_1)a}\left\{e^{-\lambda_1 x}\left[\frac{e^{-(k+\lambda_1)s}}{-k-\lambda_1}\right]_{s=0}^{s=x} + e^{\lambda_2 x}\left[\frac{e^{(-k-\lambda_2)s}}{-k-\lambda_2}\right]_{s=0}^{s=x}\right.$$

$$= \frac{1}{(\lambda_2-\lambda_1)a}\left\{\frac{e^{-\lambda_1 x}}{-k-\lambda_1}\left(e^{(-k-\lambda_1)x}-1\right) + \frac{e^{\lambda_2 x}}{-k-\lambda_2}\left(e^{(-k-\lambda_2)x}-1\right)\right\}$$

$$= \frac{1}{(\lambda_2-\lambda_1)a}\left\{\frac{\overset{\nearrow 0}{e^{-kx}}\,\overset{\nearrow 0}{e^{-\lambda_1 x}}}{k+\lambda_1} + \frac{\overset{\nearrow 0}{e^{-kx}} - \overset{\nearrow 0}{e^{\lambda_2 x}}}{-k-\lambda_2}\right\}$$

$k>0$

$e^{kx} \xrightarrow{x\to\infty} 0$

Συνεπως $\lim_{x\to\infty} y_p(x) = 0$

τα στοιχεια του βασικου συνολου οταν $x\to\infty$ $\lim_{+\infty} e^{\lambda_1 x} = 0$, $\lim_{+\infty} e^{\lambda_2 x} = 0$

Πιχύνεται οταν $k=\lambda_1$ ή $k=\lambda_2$ (*) τοτε το αντιστοιχο κομματι πασι στο μηδεν.

$\Delta<0$

Εξισώσεις Euler:

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \dots + a_1 x y' + a_0 y \quad a_0, \dots, a_n = \text{σταθερές}$$

$a_n \neq 0$, $x \neq 0$

$x > 0$

$x > 0 \to t = \log x$

$$y' = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt} \Rightarrow xy' = \frac{dy}{dt}$$

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}(y') = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dt}\left(\frac{dy}{dt}\right)\cdot\underbrace{\left(\frac{dt}{dx}\right)}_{\frac{1}{x}}$$

$$= -\frac{1}{x^2}\left(-\frac{dy}{dt} + \frac{d^2y}{dt^2}\right) \Rightarrow x^2y'' = -\frac{dy}{dt} + \frac{d^2y}{dt^2}$$

$$y''' = \frac{d}{dx}(y'') = \frac{d}{dx}\left\{\frac{1}{x^2}\left[-\frac{dy}{dt} + \frac{d^2y}{dt^2}\right]\right\} = -\frac{2}{x^3}\left[-\frac{dy}{dt} + \frac{d^2y}{dt^2}\right]$$

$$+ \frac{1}{x^2}\frac{d}{dx}\left[-\frac{dy}{dt} + \frac{d^2y}{dt^2}\right] = -\frac{2}{x^3}\left[-\frac{dy}{dt} + \frac{d^2y}{dt^2}\right] + \frac{1}{x^2}\frac{d}{dt}\left[-\frac{dy}{dt} + \frac{d^2y}{dt^2}\right]\frac{dt}{dx}$$

$$\Rightarrow x^3y''' = 2\frac{dy}{dt} - 2\frac{d^2y}{dt^2} - \frac{d^2y}{dt^2} + \frac{d^3y}{dt^3}$$

Παράδειγμα:

$$x^3y''' - x^2y'' - 2xy' - 4y = 0, \quad x > 0$$

$$\frac{d^3y}{dt^3} - \frac{3d^2y}{dt^2} + \frac{2dy}{dt} - \left[-\frac{dy}{dt} + \frac{d^2y}{dt^2}\right] - 2\frac{dy}{dt} - 4y = 0$$

$$\frac{d^3y}{dt^3} - \frac{4d^2y}{dt^2} + \frac{dy}{dt} - 4y = 0$$

$\chi.\pi \quad p(\lambda) = \lambda^3 - 4\lambda^2 + \lambda - 4 = 0$

$(\lambda-4)(\lambda^2+1) = 0$

ΒΣΛ $\{e^{4t}, \cos t, \sin t\}$, $t\in\mathbb{R}$

$y_1(x) = e^{4\log x}$, $y_2 = \cos\log x$, $y_3 = \sin\log x$

ΒΣΛ $\{x^4, \cos(\log x), \sin(\log x)\}$

Άσκηση σε σελ 113

$x^2y'' - xy' + y = 0, \quad x>0$

$t = \log x$

$$\frac{dy}{dt} + \frac{d^2y}{dt^2} - \frac{dy}{dt} + y = 0 \Rightarrow \frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = 0$$

$X.\Pi: \ p(\lambda) = \lambda^2 - 2\lambda + 1 = 0$

$\Rightarrow (\lambda-1)^2 = 0 \Rightarrow \lambda = 1$

$y_1(t) = e^t$, $y_2(t) = te^t$, $t\in\mathbb{R}$

$y_1(x) = e^{\log x} = x$, $y_2(x) = \log x e^{\log x} = x\cdot\log x$, $x>0$

iii) $(x-2)^2y'' - (x-2)y' + y = 0$

$\underset{\omega}{\vee}$

$t = \log\omega \cdots \Rightarrow y_1(\omega) = e^{\log\omega} = \omega$

$y_2(\omega) = \log\omega\, e^{\log\omega} = \omega\cdot\log\omega$

Άρα $y_1(x) = (x-2)$

$y_2(x) = (x-2)\log(x-2)$, $x>2$

Άσκηση:

$$(\sin^2 x)y'' + (tgx)y' + k^2(\cos^2 x)y = 0, \quad x \in (0, \tfrac{\pi}{2}), \quad k > 0$$

μετασχηματισμός $t = \sin x$.

$$y' = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \cos x \frac{dy}{dt}$$

$$y'' = \frac{d}{dx}(y') = \frac{d}{dx}\left(\cos x \frac{dy}{dt}\right) = -\sin x \frac{dy}{dt} + \cos x \frac{d}{dx}\left(\frac{dy}{dt}\right)$$

$$= -\sin x \frac{dy}{dt} + \cos x \frac{d}{dt}\left[\frac{dy}{dt}\right] \cdot \frac{dx}{dt}$$

$$\Rightarrow y'' = -\sin x \frac{dy}{dt} + \cos^2 x \frac{d^2 y}{dt^2}$$

Άρα έχουμε:

$$-\sin^3 x \frac{dy}{dt} + \sin^2 x \cos^2 x \frac{d^2 y}{dt^2} + \frac{\sin x}{\cos x} \cdot \cos x \frac{dy}{dt} + k^2(1 - \sin^2 x)y = 0$$

$$\Rightarrow t^2 \frac{d^2 y}{dt^2} + t \frac{dy}{dt} + k^2 y = 0, \quad t \in (0, 1)$$

Θέτω $\omega = \log t$

$$\frac{d^2 y}{d\omega^2} + k^2 \frac{dy}{d\omega} = 0$$

$$B\Sigma\eta = \{\cos k\omega, \sin k\omega\}$$

Άρα $y_0(x) = c_1 \cos(k \cdot \log(\sin x)) + c_2 \sin(k \cdot \log(\sin x))$

B51, B53, B56.